CANONICAL REPRESENTATIONS

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These lecture notes provide an introduction to the theory of so-called canonical representations, a special type of (reducible) unitary representations. The simplest way to define them and to see their importance is done in the context of the group $\mathrm{SL}(2,\mathbb{R})$, the group of 2×2 matrices of determinant one, see [16]. We have chosen a class of groups, namely $G = \mathrm{SU}(1,n)$, $n \geq 1$. Notice that $\mathrm{SL}(2,\mathbb{R})$ is isomorphic to $\mathrm{SU}(1,1)$. Canonical representations can be seen, generally speaking, as the completion of $L^2(G/K)$ with respect to a new G-invariant innner product, in the same spirit as the "complementary series" is obtained from the "principal series" for G. Here $K = \mathrm{SU}(n)$. But this is only one (but important) point of view, see section 5. Canonical representations occur also when studying tensor products of holomorphic and anti-holomorphic discrete series representations. This is explained in section 4. The connection with quantization in the sense of Berezin is not treated in these notes because we will emphasize the representation theory. On the other hand, Berezin has made a large contribution to the understanding of canonical representations.

The main problem is to decompose the canonical representations into irreducible constituents. This is not an easy task. It has been done by Berezin [1] and, later, by Upmeier and Unterberger [15]. There are however, in both treatments, conditions on the set of parameters of the representations: only large parameters are allowed. For small parameters (see [3]) an interesting new phenomenon occurs: finitely many complementary series representations take part in the decomposition. We shall treat the case $G = \mathrm{SU}(1,n)$ in detail in these notes and try to illustrate all aspects of the theory of canonical representations we have mentioned.

1. SPHERICAL FOURIER ANALYSIS ON COMPLEX HYPERBOLIC SPACES

The main reference for this section is [9].

1.1. Complex hyperbolic spaces and their bounded realizations

Let $n \geq 1$. Consider on $\mathbb{C}^{n+1}$ the Hermitian form

$$[x,y] = \overline{y}_0\, x_0 - \overline{y}_1\, x_1 - \cdots - \overline{y}_n\, x_n. \tag{1.1}$$

Let $G = \mathrm{SU}(1,n)$ be the group of $(n+1)\times(n+1)$ complex matrices which preserve this form and have determinant equal to 1. The group G acts on the projective space $P_n(\mathbb{C})$ and the stabilizer of the line generated by the vector $(1,0,\ldots,0)$ is the compact subgroup $K = \mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(n))$. We call $\mathcal{X} = G/K$ a complex hyperbolic space. $\mathcal{X}$ is, in addition, a Riemannian symmetric space of the non-compact type, of rank one, and carries a complex structure, as we will see.

Let π denote the natural projection map

$$\pi : \mathbb{C}^{n+1}\backslash\{0\} \to P_n(\mathbb{C}), \tag{1.2}$$

sending each vector to the line generated by it.

The hyperbolic space $\mathcal{X}$ is the image under π of the open set

$$\{x \in \mathbb{C}^{n+1} : [x,x] > 0\}. \tag{1.3}$$

On $\mathbb{C}^n$ we have the ususal inner product

$$(x,y) = \overline{y}_1 x_1 + \cdots + \overline{y}_n x_n \tag{1.4}$$

with norm $||x|| = (x,x)^{1/2}$. Let

$$B(\mathbb{C}^n) = \{x \in \mathbb{C}^n : ||x|| < 1\}, \tag{1.5}$$

the unit ball in $\mathbb{C}^n$. The space $\mathcal{X}$ can be realized as the unit ball in $\mathbb{C}^n$: the map from (1.3) to $\mathbb{C}^n$, given by

$$x \to y \quad \text{with} \quad y_p = x_p x_0^{-1} \tag{1.6}$$

defines, by passing to the quotient space, a real analytic bijection of $\mathcal{X}$ onto $B(\mathbb{C}^n)$. G acts on $B(\mathbb{C}^n)$ transitively by fractional linear transformations. If $g \in G$ is of the form $g \in \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with matrices $a\,(1\times 1)$, $b\,(1 \times n)$, $c\,(n \times 1)$ and $d\,(n \times n)$, then

$$g \cdot y = (dy + c)\,(\langle b, y\rangle + a)^{-1} \tag{1.7}$$

where y and c are regarded as column vectors and

$$\langle b, y\rangle = b_1 y_1 + \cdots + b_n y_n. \tag{1.8}$$

Clearly $K = \text{Stab}\,(o)$.

An easy computation shows that

$$1 - (g \cdot y,\, g \cdot z) = \overline{(\langle b, z\rangle + a)}^{-1} \cdot [1 - (y, z)] \cdot (\langle b, y\rangle + a)^{-1}, \tag{1.9}$$

$$1 - \|g \cdot y\|^2 = [1 - \|y\|^2] \cdot |\langle b, y\rangle + a|^{-2} \tag{1.10}$$

and

$$\langle b, y\rangle + a = (\langle \tilde{b},\, g \cdot y\rangle + \tilde{a})^{-1} \text{ if } g^{-1} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}. \tag{1.11}$$

On the other hand, the absolute value of the Jacobian of the real analytic transformation $y \to g \cdot y$ ($y \in B(\mathbb{C}^n)$) is easily seen to be equal to

$$|\langle b,\, y\rangle + a|^{-2(n+1)}. \tag{1.12}$$

If dy is the Euclidean measure on $\mathbb{C}^n$, then clearly

$$d\mu(y) = (1 - \|y\|^2)^{-(n+1)} dy \tag{1.13}$$

is a G-invariant measure on $B(\mathbb{C}^n)$.

1.2. Fine structure of $\mathrm{SU}(1, n)$

Let J be the $(n+1) \times (n+1)$ matrix $\text{diag}\,\{1, -1, \ldots, -1\}$. For any complex matrix X of type $(n+1) \times (n+1)$ we set $X^* = J\,\overline{X}^t J$.

The Lie algebra $\mathfrak{g}$ of G consists of the matrices X that verify the relation

$$X + X^* = 0, \ \text{trace}\, X = 0. \tag{1.14}$$

These are the matrices of the form

$$\begin{pmatrix} Z_1 & Z_2 \\ \overline{Z}_2^t & Z_3 \end{pmatrix} \tag{1.15}$$

with Z_1 and Z_3 anti-Hermitian and Z_2 arbitrary, $\text{trace}\,(Z_1 + Z_3) = 0$. Let θ be the involutive automorphism of $\mathfrak{g}$ defined by

$$\theta X = JXJ. \tag{1,16}$$

Then θ is a Cartan involution with the usual decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Here $\mathfrak{k}$ is the Lie algebra of K.

Let L be the following element of $\mathfrak{g}$:

$$L = \begin{pmatrix} 0 & 0 & 1 \\ 0 & O & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.17}$$

We have $L \in \mathfrak{p}$ and $\mathfrak{a} = \mathbb{R}\,L$ is a maximal Abelian subspace of $\mathfrak{p}$. We are going to diagonalize the operator $\text{ad}\,L$. The centralizer of L in $\mathfrak{k}$ is

$$\mathfrak{m} = \left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & u \end{pmatrix} \ : \ u + \overline{u} = 0,\ v \in \mathfrak{u}(n-1),\ 2u + \text{trace}\, v = 0 \right\}. \tag{1.18}$$

Let $\alpha = 1$. The nonzero eigenvalues of $\operatorname{ad} L$ are $\pm\alpha$, $\pm 2\alpha$. The space $\mathfrak{g}_\alpha$ consists of the matrices

$$X = \begin{pmatrix} 0 & z^* & 0 \\ z & O & -z \\ 0 & z^* & 0 \end{pmatrix} \tag{1.19}$$

where z is a matrix of type $(n-1, 1)$ and $z^* = -\overline{z}^t$.

The dimension of $\mathfrak{g}_\alpha$ is equal to $m_\alpha = 2(n-1)$. The space $\mathfrak{g}_{2\alpha}$ consists of the matrices of the form

$$X = \begin{pmatrix} w & 0 & -w \\ 0 & O & 0 \\ w & 0 & -w \end{pmatrix} \tag{1.20}$$

with $w + \overline{w} = 0$. The dimension of $\mathfrak{g}_{2\alpha}$ is equal to $m_{2\alpha} = 1$. We have $\mathfrak{g} = \mathfrak{g}_{-2\alpha} + \mathfrak{g}_{-\alpha} + \mathfrak{a} + \mathfrak{m} + \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$.

Let A be the subgroup $\exp \mathfrak{a}$. This is the subgroup of the matrices

$$a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \tag{1.21}$$

where t is a real number. The centralizer of A in K is the subgroup M of the matrices

$$\begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & u \end{pmatrix} \tag{1.22}$$

with $|u| = 1$ and $v \in \mathrm{U}(n-1)$, $u^2 \det v = 1$. The Lie algebra of M is $\mathfrak{m}$. The subspace $\mathfrak{n} = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ is a nilpotent subalgebra. Set $N = \exp \mathfrak{n}$. This is the subgroup of the matrices

$$n(z, w) = \begin{pmatrix} 1 + w - \frac{1}{2}[z, z] & z^* & -w + \frac{1}{2}[z, z] \\ z & I & -z \\ w - \frac{1}{2}[z, z] & z^* & 1 - w + \frac{1}{2}[z, z] \end{pmatrix} \tag{1.23}$$

with $w + \overline{w} = 0$ and with z a matrix of type $(n-1, 1)$, $z^* = -\overline{z}^t$, and if

$$z = \begin{pmatrix} z_2 \\ \vdots \\ z_n \end{pmatrix}, \quad z' = \begin{pmatrix} z'_2 \\ \vdots \\ z'_n \end{pmatrix},$$

then $[z, z'] = -\overline{z}'_2 z_2 - \cdots \overline{z}'_n z_n$.

The composition law in N is the following:

$$n(w, z) \cdot n(w', z') = n(w + w' + \operatorname{Im}[z, z'], z + z'). \tag{1.24}$$

The subgroup A normalizes N:

$$a_t\, n(z, w)\, a_{-t} = n(e^{2t} w, e^t z). \tag{1.25}$$

Let 2ρ be the trace of the restriction of $\operatorname{ad} L$ to $\mathfrak{n}$:

$$\rho = \frac{1}{2}(m_\alpha + 2m_{2\alpha}) = n. \tag{1.26}$$

We have the Iwasawa decomposition $G = KAN = NAK$. Each $g \in G$ can uniquely be written as $g = k a_{t(g)} n$ accordingly. One has the corresponding integral formula:

$$\int_G f(g)\, dg = \int_{KAN} f(k a_t n)\, e^{2\rho t} dk\, dt\, dn \tag{1.27}$$

for $f \in \mathcal{D}(G)$. This is also equal to

$$\int_{NAK} f(na_t k)\, \mathrm{e}^{-2\rho t}\, dndtdk. \tag{1.28}$$

Here $dn = dzdw$ $(n = n(z,w))$ and dk is the normalized Haar measure on K. Observe that NA parametrizes $\mathcal{X} = G/K$. Moreover, we have the Cartan decomposition $G = KA_+K$ where

$$A_+ = \{a_t \,:\, t \geq 0\}, \tag{1.29}$$

and, after dg is normalized according to (1.27), the corresponding integral formula

$$\int_G f(g)\, dg = \int_K \int_0^\infty \int_K f(ka_t k')\, \delta(t)\, dkdtdk'. \tag{1.30}$$

Here

$$\delta(t) = 2\frac{\pi^n}{\Gamma(n)}(\sinh t)^{m_\alpha}\,(\frac{\sinh 2t}{2})^{m_{2\alpha}}. \tag{1.31}$$

1.3. Spherical functions, inversion and Plancherel formula

For $s \in \mathbb{C}$ let

$$\varphi_s(g) = \int_K \mathrm{e}^{(s-\rho)t(g^{-1}k)} dk \quad (g \in G) \tag{1.32}$$

be the zonal spherical function with parameter s, in integral form, according to Harish-Chandra. It is known that $\varphi_s(g) = \varphi_{-s}(g) = \varphi_s(g^{-1})$. Furthermore let $c(s)$ denote Harish-Chandra's c-function:

$$c(s) = \Gamma(n)\, 2^{\rho - s}\frac{\Gamma(s)}{\Gamma(\frac{s+\rho}{2})^2}. \tag{1.33}$$

For $f \in \mathcal{D}(G//K)$, the space of bi-K-invariant, compactly supported C^∞-functions on G, we define its spherical Fourier transform as

$$\widehat{f}(s) = \int_G f(g)\, \varphi_{-s}(g)\, dg \quad (s \in \mathbb{C}). \tag{1.34}$$

$\widehat{f}$ is a function of Paley-Wiener class, and $\widehat{f}$ is even in the argument s. One has:

Inversion formula:

$$f(g) = c_0 \int_0^\infty \widehat{f}(i\mu)\, \varphi_{i\mu}(g)\, \frac{d\mu}{|c(i\mu)|^2} \qquad (g \in G), \tag{1.35}$$

and

Plancherel formula:

$$\int_G |f(g)|^2\, dg = c_0 \int_0^\infty |\widehat{f}(i\mu)|^2\, \frac{d\mu}{|c(i\mu)|^2}, \tag{1.36}$$

where $c_0 = 2^{2n-2}\,\Gamma(n)/\pi^{n+1}$.

The function $\varphi(t,s) := \varphi_s(a_t)$ is the unique solution of the ordinary differential equation

$$\frac{d^2y}{dt^2} + [m_\alpha \frac{\cosh t}{\sinh t} + 2m_{2\alpha}\frac{\cosh 2t}{\sinh 2t}]\frac{dy}{dt} = (s^2 - \rho^2)y, \tag{1.37}$$

that satisfies $\varphi(0,s) = 1$. So

$$\varphi(t,s) = {}_2F_1(\frac{s+\rho}{2}, \frac{-s+\rho}{2}; \rho; -\sinh^2 t). \tag{1.38}$$

There is another solution for $t > 0$, $\Phi(t,s)$, which has the asymptotic behaviour $\mathrm{e}^{(s-\rho)t}$ as $t \to \infty$ and is given explicitly by

$$\Phi(t,s) = 2^{s-\rho}\,(\sinh t)^{s-\rho}\,{}_2F_1(\frac{-s-\rho+2}{2}, \frac{-s+\rho}{2}; 1-s; -\sinh^{-2} t) \tag{1.39}$$

for $s \neq 1, 2, 3, \ldots$. If s is not an integer, then

$$\varphi(t,s) = c(s)\,\Phi(t,s) + c(-s)\,\Phi(t,-s) \quad (t > 0). \tag{1.40}$$

Moreover, as t approaches 0, $\Phi(t,s)$ and $\frac{\partial\Phi}{\partial t}(t,s)$ have the following asymptotic behaviour:

$$\Phi(t,s) \sim \begin{cases} C(s)\,t^{2-2n} & \text{if } n \neq 1 \\ C(s)\log t & \text{if } n = 1 \end{cases}, \tag{1.41}$$

$$\frac{\partial\Phi(s,t)}{\partial t} \sim \begin{cases} (2-2n)\,C(s)\,t^{1-2n} & \text{if } n \neq 1 \\ C(s)\,1/t & \text{if } n = 1, \end{cases} \tag{1.42}$$

for a certain function C of s. Notice that Φ and $\frac{\partial\Phi(s,t)}{\partial t}$ are integrable with respect to the measure $\delta(t)dt$ on $(0,\infty)$ whenever $\operatorname{Re} s < -\rho$.

It easily follows from (1.36) that for $t > 0$:

$$f(a_t) = c_0 \int_{-\infty}^{\infty} \widehat{f}(i\mu)\,\Phi(t,-i\mu)\,\frac{d\mu}{c(i\mu)}. \tag{1.43}$$

Since $\widehat{f}$ is of Paley-Wiener class and $c(s)^{-1}$ of polynomial growth (see (1.33)) for $\operatorname{Re} s > -1$, one has in addition, by Cauchy's theorem:

$$f(a_t) = c_0 \int_{-\infty}^{\infty} \widehat{f}(\sigma + i\mu)\,\Phi(t,-\sigma - i\mu)\,\frac{d\mu}{c(\sigma + i\mu)} \tag{1.44}$$

for $\sigma > -1$, $t > 0$ and $f \in \mathcal{D}(G//K)$.

2. CANONICAL REPRESENTATIONS

2.1. Definition of canonical representations

For $\lambda \in \mathbb{R}$ and $g \in G$ we set

$$\psi_\lambda(g) = (1 - \|y\|^2)^\lambda \tag{2.1}$$

where $y \in B(\mathbb{C}^n)$, $y = g \cdot o$. Clearly ψ_λ is a bi-K-invariant continuous function on G. Observe that $\psi_\lambda(a_t) = (\cosh t)^{-2\lambda}$ $(t \in \mathbb{R})$. An easy computation shows that

$$\psi_\lambda(g_1^{-1}g_2) = \left\{ \frac{(1-\|y\|^2)\,(1-\|z\|^2)}{[1-(y,z)]\,[1-(z,y)]} \right\}^\lambda \tag{2.2}$$

if $z, y \in B(\mathbb{C}^n)$, $z = g_1 \cdot o$, $y = g_2 \cdot o$.

Let us denote this expression by $B_\lambda(y,z)$. B_λ is called a *Berezin kernel* of $\mathcal{X}$. Since products and (uniform) limits of positive-definite kernels are again positive-definite, we easily get, by expanding $[1-(z,y)]^{-\lambda}$ into a power series:

$$[1-(z,y)]^{-\lambda} = \sum_{m=0}^{\infty} \binom{-\lambda}{m} (z,y)^m\,(-1)^m \tag{2.3}$$

with $\binom{-\lambda}{m} = \frac{(-\lambda)(-\lambda-1)\ldots(-\lambda-m+1)}{m!}$, that B_λ is a positive-definite kernel for $\lambda \geq 0$. Or, otherwise said, ψ_λ is a positive-definite function for $\lambda \geq 0$.

Let π_λ denote the unitary representation of G naturally associated with ψ_λ or B_λ.

We call the π_λ $(\lambda \geq 0)$ *canonical representations* after Vershik, Gel'fand and Graev [16] and we shall study in this section their spectral decomposition in detail.

2.2. Spectral decomposition

The function ψ_λ is the reproducing distribution of π_λ in the sense of L. Schwartz, see [2]. We shall determine the integral decomposition of ψ_λ into (elementary) positive-definite spherical functions. It is well-known (see [11]) that the functions φ_s, defined in (1.32), are positive-definite if and only if s is purely imaginary or $-\rho \leq s \leq \rho$.

By (1.13), the function ψ_λ is actually a function in $L^1(G)$ for $\lambda > \rho$, or even for $\operatorname{Re}\lambda > \rho$, since ψ_λ is well-defined for complex λ too. So in this case ($\operatorname{Re}\lambda > \rho$), it suffices to determine the spherical Fourier transform $a_\lambda(\mu)$ of ψ_λ:

$$a_\lambda(\mu) = \int_G \psi_\lambda(g)\,\varphi_{-i\mu}(g)\,dg. \tag{2.4}$$

The computation of $a_\lambda(\mu)$ is surprisingly simple. Applying the Cartan decomposition $G = KA_+K$, (1.30) and (1.39), we get by making the change of variable $x = \sinh^2 t$,

$$a_\lambda(\mu) = \frac{\pi^n}{\Gamma(n)} \int_0^\infty {}_2F_1\left(\frac{-i\mu+\rho}{2}, \frac{i\mu+\rho}{2}; \rho; -x\right)(1+x)^{-\lambda}\, x^{n-1} dx. \tag{2.5}$$

This expression is by [7], 20.2 (9) equal to

$$a_\lambda(\mu) = \pi^n \frac{\Gamma(\lambda + \frac{i\mu-\rho}{2})\,\Gamma(\lambda + \frac{-i\mu-\rho}{2})}{\Gamma(\lambda)^2}. \tag{2.6}$$

We may, in particular, reconclude that ψ_λ is a positive-definite function for $\lambda > \rho$.

Moreover

$$(\psi_\lambda, f) = \int_G \psi_\lambda(g)\,f(g)\,dg = c_0 \int_0^\infty a_\lambda(\mu)\,\widehat{f}(i\mu)\,\frac{d\mu}{|c(i\mu)|^2} \tag{2.7}$$

for all $f \in \mathcal{D}(G//K)$. Here c_0 is as in (1.35).

We will now describe the decomposition (2.7) in another way, in order to gain insight how to proceed in the case $0 < \lambda \leq \rho$, where ψ_λ is still positive-definite.

We apply (1.44). The function Φ satisfies:

$$|\Phi(t, -\sigma - i\mu)||\psi_\lambda(t)|\delta(t) \leq C_0 e^{-(\sigma + 2\operatorname{Re}\lambda - \rho)t} \tag{2.8}$$

for some positive constant C_0 and t large. Thus for σ and λ such that $\sigma + 2\operatorname{Re}\lambda > \rho$:

$$(\psi_\lambda, f) = c_0 \int_{-\infty}^\infty \widehat{f}(\sigma + i\mu)\,b_\lambda(\sigma + i\mu)\,\frac{d\mu}{c(\sigma + i\mu)} \tag{2.9}$$

where

$$b_\lambda(s) := \int_0^\infty \Phi(t, -s)\,\psi_\lambda(t)\,\delta(t)\,dt. \tag{2.10}$$

For $\operatorname{Re}\lambda > \rho$ we have:

$$\begin{aligned} a_\lambda(\mu) &= \int_0^\infty \psi_\lambda(a_t)\,[c(i\mu)\Phi(t, i\mu) + c(-i\mu)\Phi(t, -i\mu)]\delta(t)dt \\ &= c(i\mu)b_\lambda(-i\mu) + c(-i\mu)b_\lambda(i\mu). \end{aligned} \tag{2.11}$$

We will now take a closer look at the function b_λ. Since $\Phi(t, -s)$ is analytic in s for $\operatorname{Re}(s) > -1$, it is immediately seen that $b_\lambda(s)$ is analytic in s on $V_\lambda = \{s \mid \operatorname{Re}(s) > \max(\rho - 2\operatorname{Re}\lambda, -1)\}$. We will consider the problem of analytic continuation of $b_\lambda(s)$.

Fix $\lambda > 0$ and let $C(s) = 2^{s-\rho}\pi^n/\Gamma(n)$. Using (1.39) and making the substitutions $x = \sinh^{-2}t$ and $x = y/(1-y)$, we get

$$\begin{aligned} b_\lambda(s) &= C(s) \int_0^\infty {}_2F_1\left(\frac{s-\rho+2}{2}, \frac{s+\rho}{2}; 1+s; -x\right) x^{\frac{s-\rho}{2}+\lambda-1}(1+x)^{-\lambda} dx \\ &= C(s) \int_0^1 {}_2F_1\left(\frac{s-\rho+2}{2}, \frac{s+\rho}{2}; 1+s; \frac{y}{1-y}\right) y^{\frac{s-\rho}{2}+\lambda-1}(1-y)^{\frac{-s+\rho-2}{2}} dy. \end{aligned} \tag{2.12}$$

Applying the relation

$${}_2F_1(a, b; c; z) = (1-z)^{-a}\,{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) \tag{2.13}$$

(cf. [8], 2.1.4 (p. 64)) yields

$$b_\lambda(s) = C(s) \int_0^1 {}_2F_1\left(\frac{s-\rho+2}{2}, \frac{s-\rho+2}{2}; 1+s; y\right) y^{\frac{s-\rho}{2}+\lambda-1} dy. \tag{2.14}$$

By substituting the series expansion

$${}_2F_1(a,b;c;z) = \sum_{l=0}^{\infty} \frac{(a)_l\,(b)_l}{(c)_l\, l!} z^l \quad (|z|<1) \tag{2.15}$$

and taking care of the possible singularity in $y=1$ (in case $n=1$) we obtain

$$b_\lambda(s) = C(s) \sum_{l=0}^{\infty} \frac{(\frac{s-\rho}{2}+1)_l^2}{(s+1)_l\, l!} \cdot \frac{1}{\frac{s-\rho}{2}+\lambda+l}. \tag{2.16}$$

This series is absolutely convergent for $s \neq -1, -2, \ldots$ and $s \neq s_l(\lambda) = \rho - 2\lambda - 2l$ ($l = 0, 1, 2, \ldots$), since the terms are majorated by $l^{-1-\delta}$ for some δ between 0 and n, for l large. This can easily be seen by using Euler's limit formula for the gamma function:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{(z)_{n+1}}. \tag{2.17}$$

Thus b_λ has a meromorphic extension to $\mathbb{C}$ with poles in $s = s_l(\lambda)$ ($l = 0, 1, 2, \ldots$) and $s = -1, -2, -3, \ldots$. The residues in $s_l(\lambda)$ are equal to

$$\frac{2^{2\lambda+2l-2\rho+1}\,\pi^n}{\Gamma(n)} \cdot \frac{(1-\lambda-l)_l^2}{(\rho-2\lambda-2l+1)_l\, l!}, \tag{2.18}$$

if $s_l(\lambda) \neq -1, -2, -3, \ldots$. An easy observation shows that these residues are strictly positive for $\lambda > 0$ for all values of l such that $s_l > 0$.

Consider the relation (2.11) again. The explicit expression for a_λ shows that for any fixed $\mu \neq 0$ in $\mathbb{R}$, $\lambda \to a_\lambda(\mu)$ depends analytically on λ for λ in some strip around the positive real axis. So does the right-hand side of (2.11). Thus this relation actually holds for all $\mu \neq 0$ in $\mathbb{R}$ and $\lambda > 0$.

Let Φ_{-s} be the bi-K-invariant function on G defined by $\Phi_{-s}(a_t) := \Phi(t, -s)$ for $t > 0$. The definition of b_λ can then be reformulated as

$$b_\lambda(s) = \int_{\mathcal{X}} \psi_\lambda(x)\, \Phi_{-s}(x)\, dx \tag{2.19}$$

for $s \in V_\lambda$. Formula (1.30) together with (1.41), (1.42) shows that the integral exists for those s. Let us define the differential operator Δ_λ on $\mathcal{X} = G/K$ by

$$\Delta_\lambda := c_\lambda\, (\Delta + d_\lambda) \tag{2.20}$$

with $c_\lambda = -1/(4\lambda^2)$, $d_\lambda = -4\lambda(\lambda-\rho)$ and Δ the Laplace-Beltrami operator of $\mathcal{X}$.

A direct computation using the explicit form for ψ_λ yields

$$\Delta_\lambda \psi_\lambda = \psi_{\lambda+1} \tag{2.21}$$

for all $\lambda > 0$. Fix $\lambda > 0$. One has

$$\int_{\mathcal{X}} \Delta\psi_\lambda(x)\, \Phi_{-s}(x)\, dx$$
$$= C_0(s) + \int_{\mathcal{X}} \psi_\lambda(x) \Delta\Phi_{-s}(x)\, dx = C_0(s) + (s^2-\rho^2) b_\lambda(s) \tag{2.22}$$

for all $s \in \mathbb{C}$ such that $\mathrm{Re}\,(s) > \rho - 2\lambda$ with

$$C_0(s) := \lim_{t\downarrow 0} \psi_\lambda(a_t)\, \frac{\partial\Phi}{\partial t}(t, -s)\, \delta(t). \tag{2.23}$$

This limit can easily be computed for $\mathrm{Re}(s) > 0$ (cf. [9], Ch. IV, section V, §2, p. 415-416) and is equal to

$$C_0(s) = -\frac{2^{2-2\rho}\pi^n}{\Gamma(n)} \cdot sc(s). \tag{2.24}$$

For $\lambda > 0$ and $s \in V_\lambda \cap \{\mathrm{Re}(s) > 0\}$ we obtain

$$\begin{aligned} b_{\lambda+1}(s) &= \int_{\mathcal{X}} \Delta_\lambda \psi_\lambda(x)\, \Phi_{-s}(x)\, dx \\ &= c_\lambda \{C_0(s) + (s^2 - (\rho - 2\lambda)^2)\, b_\lambda(s)\}. \end{aligned} \tag{2.25}$$

Because $C_0(s)$ is analytic for $\mathrm{Re}(s) > -1$, this relation can be used to extend b_λ to $\mathrm{Re}(s) > -1$ by iteration. Since $c(s)$ and $b_{\lambda+k}(s)$ remain bounded as $|s| \to \infty$ in the strip $0 \le \mathrm{Re}(s) \le \rho$ for sufficiently large $k \in \mathbb{N}$, it is now easily seen that $b_\lambda(s)$ remains bounded too when $|s| \to \infty$ in that strip. We thus have:

Proposition 2.1. *Let $\lambda > 0$. The function $b_\lambda(s)$, defined for $s \in V_\lambda$, has a meromorphic extension to $\mathbb{C}$ with poles in $s_l(\lambda) = \rho - 2\lambda - 2l$ $(l = 0, 1, 2, \ldots)$ and $-1, -2, -3, \ldots$, given by (2.16). The residues in $s_l(\lambda)$ are equal to*

$$\frac{2^{2\lambda+2l-2\rho+1}\pi^n}{\Gamma(n)} \, \frac{(1-\lambda-l)_l^2}{(\rho - 2\lambda - 2l + 1)_l\, l!},$$

provided $s_l(\lambda) \ne -1, -2, -3, \ldots$. Moreover, b_λ remains bounded as $|s| \to \infty$ in the strip $0 \le \mathrm{Re}(s) \le \rho$.

Fix $\lambda > 0$. Let $f \in \mathcal{D}(G//k)$ and consider the function

$$g_\lambda : s \to c_0 \, \frac{\widehat{f}(s) b_\lambda(s)}{c(s)}.$$

The function g_λ is meromorphic for $\mathrm{Re}(s) \ge 0$ with simple poles in $s_l = s_l(\lambda)$, l such that $s_l > 0$. Let γ_R be the contour determined by the rectangle given by the points $\pm iR$ and $\rho \pm iR$. Since $\widehat{f}$ is of Paley-Wiener type and b_λ remains bounded as $|s| \to \infty$ in the strip $0 \le \mathrm{Re}(s) \le \rho$, integrating g_λ over γ_R and letting R tend to infinity yields

$$(\psi_\lambda, f) = 2\pi \sum_{l, s_l > 0} r_l(\lambda)\, (\varphi_{s_l}, f) + c_0 \int_{-\infty}^{\infty} \frac{b_\lambda(i\mu)}{c(i\mu)}\, (\varphi_{i\mu}, f)\, d\mu \tag{2.26}$$

for $\lambda > 0$. Here we used relation (2.9) with $\sigma = \rho$ and

$$r_l(\lambda) := \frac{1}{c_0\, c(s_l)} \mathrm{Res}_{s=s_l}\, b_\lambda(s). \tag{2.27}$$

Thus we finally obtain, by using (2.11),

$$(\psi_\lambda, f) = 2\pi \sum_{l, s_l > 0} r_l(\lambda)\, (\varphi_{s_l}, f) + c_0 \int_0^{\infty} a_\lambda(i\mu)\, (\varphi_{i\mu}, f)\, \frac{d\mu}{|c(i\mu)|^2} \tag{2.28}$$

for all $\lambda > 0$. So, in particular, we pick up complementary series representations in $s = s_l(\lambda)$.

Theorem 2.2. *Let $\lambda > 0$. If $\lambda \ge \rho/2$, π_λ decomposes into a direct integral of principal series representations. If $0 < \lambda < \rho/2$ the spectrum of π_λ has a discrete part consisting of finitely many complementary series representations. The continuous part consists of principal series representations.*

3. ASYMPTOTIC BEHAVIOUR OF THE CANONICAL REPRESENTATIONS

In this section we consider the asymptotic behaviour of π_λ (or ψ_λ) as λ tends to infinity. Therefore we apply an alternative meaning of ψ_λ. We refer to [10].

Consider on $\{x \in \mathbb{C}^{n+1} : [x, x] > 0\}$ the Riemannian metric

$$ds^2 = -\frac{[dx, dx]}{[x, x]}. \tag{3.1}$$

This metric is invariant under $x \to \lambda x$ ($\lambda \in \mathbb{C}$, $\lambda \neq 0$) and thus gives a Riemannian metric on $\mathcal{X}$ which is invariant under G. Corresponding to this metric we have a G-invariant second order differential operator, the Laplacian Δ, which we already met in section 2. Let π be the map defined in (1.2). If f is a function of class C^2 on $\mathcal{X}$, we set $\widetilde{f} = f \circ \pi$, so that $\widetilde{f}$ is defined on the open set

$$\{x \in \mathbb{C}^{n+1} \ : \ [x,x] > 0\}$$

and satisfies $\widetilde{f}(\lambda x) = \widetilde{f}(x)$ ($\lambda \in \mathbb{C}$, $\lambda \neq 0$). We have

$$\widetilde{\Delta f} = [x,x]\widetilde{\Delta}\widetilde{f} \tag{3.2}$$

where $\widetilde{\Delta}$ is the pseudo-Laplacian associated with the pseudo-Euclidean metric $ds^2 = -[dx, dx]$ on $\mathbb{C}^{n+1}$.

Consider also on the set $\{x \in \mathbb{C}^{n+1} \ : \ [x,x] > 0\}$ the function $\widetilde{Q}$ defined by

$$\widetilde{Q}(x) = \frac{|x_0|^2}{[x,x]}. \tag{3.3}$$

$\widetilde{Q}$ satisfies $\widetilde{Q}(tx) = \widetilde{Q}(x)$ ($t \in \mathbb{C}$, $t \neq 0$) and therefore $\widetilde{Q} = Q \circ \pi$ for some function Q on $\mathcal{X}$. Q has the following properties:

- Q is invariant under K,
- Q is real analytic,
- $Q(x) \geq 1$,
- Q has a non-degenerate critical point $x^0 = eK$; the Hessian of Q at x^0 has signature $(2n, 0)$,
- $Q(x) = t$ $(t > 1)$ is a K-orbit on $\mathcal{X}$.

Let F be a complex-valued function on $\mathbb{R}$ of class C^2. Then

$$\Delta(F \circ Q) = (LF) \circ Q \tag{3.4}$$

where L is the ordinary differential operator

$$L = a(t)\frac{d^2}{dt^2} + b(t)\frac{d}{dt}$$

with $a(t) = 4t(t-1)$, $b(t) = 4[(n+1)t - 1]$. This follows easily from (3.2).

Recall that $\mathcal{D}(\mathcal{X})$ is the space of complex-valued C^∞-functions on $\mathcal{X}$ with compact support. Fix an invariant measure dx on $\mathcal{X}$, corresponding to the Riemannian metric. If t is not a critical value of Q (so $t \neq 1$), we can define the average $M_f(t)$ of a function $f \in \mathcal{D}(\mathcal{X})$ over the surface $\{Q(x) = t\}$ by means of the formula

$$\int_{\mathcal{X}} F(Q(x))\, f(x)\, dx = \int_1^\infty F(t)\, M_f(t)\, dt \tag{3.5}$$

for any continuous function F on $\mathbb{R}$.

The function M_f has a singularity at the critical value $t = 1$ of Q. More precisely

$$M_f(t) = Y(t-1)\,(t-1)^{n-1}\,\varphi(t) \tag{3.6}$$

with $\varphi \in \mathcal{D}(\mathbb{R})$. Here Y is the Heaviside function: $Y(t) = 1$ for $t \geq 0$, $Y(t) = 0$ for $t < 0$. Moreover

$$c\, f(x^0) = \varphi(1)$$

where $c = \pi^n/\Gamma(n)$. Since Q is a K-invariant function, we can associate with Q a G-invariant kernel K_Q on $\mathcal{X} \times \mathcal{X}$, with

$$\widetilde{K}_Q(x_1, x_2) = \frac{[x_1, x_2]\,[x_2, x_1]}{[x_1, x_1]\,[x_2, x_2]} \tag{3.7}$$

$(x_1, x_2 \in \{x \in \mathbb{C}^{n+1} \ : \ [x,x] > 0\})$.

One easily verifies that this kernel corresponds on $B(\mathbb{C}^n) \times B(\mathbb{C}^n)$ to the kernel

$$\frac{[1 - (y,z)]\,[1 - (z,y)]}{[1 - \|y\|^2]\,[1 - \|z\|^2]} \tag{3.8}$$

if $x_1 \to y$, $x_2 \to z^*(y, z \in B(\mathbb{C}^n))$. Therefore, $\psi_\lambda(g) = Q(g)^{-\lambda}$ for all $g \in G$. This interpretation of ψ_λ will be of great help in finding the asymptotic behaviour of ψ_λ as $\lambda \to \infty$.

Consider the distribution

$$f \to \int_{\mathcal{X}} Q(x)^{-\lambda} f(x)\, dx$$

($f \in \mathcal{D}(\mathcal{X})$). By (3.6) we get

$$\int_{\mathcal{X}} Q(x)^{-\lambda} f(x)\, dx = \int_1^\infty t^{-\lambda} (t-1)^{n-1} \varphi(t)\, dt.$$

Observe that this expression is an entire analytic function of λ. For $k = 0, 1, 2, \ldots$ one has

$$\int_1^\infty t^{-\lambda} (t-1)^{n+k-1}\, dt = \frac{\Gamma(\lambda - n - k)\, \Gamma(n+k)}{\Gamma(\lambda)},$$

for example for $\operatorname{Re} \lambda > n + k$. Write

$$\varphi(t) = \varphi(1) + (t-1)\varphi'(1) + \frac{(t-1)^2}{2} \psi(t)$$

with $|\psi(t)| \le \max_s |\varphi^{(2)}(s)|$, and consider the distribution T_λ given by

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda - n)\, \Gamma(n) c} Q(x)^{-\lambda} = \frac{\Gamma(\lambda)}{\pi^n\, \Gamma(\lambda - n)} Q(x)^{-\lambda},$$

for $\lambda \to \infty$. We get

$$c\langle T_\lambda, f\rangle = \varphi(1) + \frac{n}{\lambda}\varphi'(1) + \frac{1}{\lambda^2}(\varphi'(1) + \frac{1}{2}\varphi''(1))\, n(n+1) + O(\frac{1}{\lambda^3}) \tag{3.9}$$

as $\lambda \to \infty$.

Let L' denote the transpose of L with respect to dt. We have $M_{\Delta^k f} = L'^k M_f$ for all $k \in \mathbb{N}$. A computation yields

$$L'[(t-1)^{n-1} \varphi(t)] = (t-1)^{n-1} \{[(4n+4)t - 4]\varphi'(t) + 4t(t-1)\varphi''(t)\}$$

and from this equation one can derive that

$$c\Delta f(x^0) = 4n\, \varphi'(1),$$
$$c\Delta^2 f(x^0) = 16\, n(n+1)\, [\varphi''(1) + \varphi'(1)].$$

It is an easy exercise to arrive at

$$\varphi'(1) = \frac{c}{4n} \Delta f(x^0), \tag{3.10}$$

$$\varphi''(1) = \frac{c}{16n(n+1)} \{\Delta^2 f(x^0) - 4(n+1)\Delta f(x^0)\}. \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9) yields:

$$\langle T_\lambda, f\rangle = f(x^0) + \frac{1}{4\lambda}\Delta f(x^0)$$

$$+ \frac{1}{32\lambda^2}\{\Delta^2 f(x^0) + 4(n+1)\Delta f(x^0)\} + O(\frac{1}{\lambda^3}) \quad (\lambda \to \infty). \tag{3.12}$$

So $T_\lambda \to \delta$ as $\lambda \to \infty$. In terms of Berezin quantization (cf. [1], where $\lambda = 1/h$ with h denoting Planck's constant) it means in particular that the *correspondence* principle is true.

It is clear that there is no obstruction in determining higher order terms of the asymptotic expansion, due to our method.

4. TENSOR PRODUCTS OF HOLOMORPHIC AND ANTI-HOLOMORPHIC DISCRETE SERIES

4.1. The space $L^2(G/K, l)$

Denote by χ_l ($l \in \mathbb{Z}$) the character of K given by

$$\chi_l : \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \to a^l \tag{4.1}$$

where $|a| = 1$, $d \in \mathrm{U}(n)$, $a \det d = 1$. Let $\rho_l = \mathrm{Ind}_{K\uparrow G}\, \chi_l$ and V_l the space of ρ_l. So $f \in V_l$ if

(i) $f : G \to \mathbb{C}$ is measurable,

(ii) $f(gk) = \chi_l(k^{-1}) f(g)$,

(ii) $\|f\|^2 = \int_{G/K} |f(g)|\, d\mu(\overline{g}) < \infty$, where $\overline{g} = gK$.

Here $d\mu(\overline{g})$ is the invariant measure on $G/K \simeq B(\mathbb{C}^n)$, see (1.13). Instead of V_l one also uses the notation $L^2(G/K, l)$. We shall identify V_l with a space of functions on the unit ball $B = B(\mathbb{C}^n)$ in $\mathbb{C}^n$. Recall that G acts on B, by (1.7), $K = \mathrm{Stab}\,(o)$ and $gK \in G/K$ corresponds to $g \cdot o$. Now define

$$Af(g) = a^l\, f(g) \tag{4.2}$$

for $f \in L^2(G/K, l)$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $Af(gk) = Af(g)$ for all $k \in K$. So Af is defined on B and one has

$$\|f\|^2 = \int_B |Af(z)|^2\, (1 - \|z\|^2)^l\, d\mu(z),$$

with $d\mu(z)$ as in (1.13). Let $\mathcal{H}_l$ denote the Hilbert space of all measurable functions φ on B such that

$$\int_B |\varphi(z)|^2\, (1 - \|z\|^2)^l\, d\mu(z) < \infty. \tag{4.3}$$

$\mathcal{H}_l$ is a G-space; G acts unitarily in $\mathcal{H}_l$ by π_l, given by

$$\pi_l(g)\varphi(z) = \varphi(g^{-1} \cdot z)\, (\langle b, z\rangle + a)^{-l}$$

if $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. A is a unitary intertwining operator between ρ_l and π_l.

4.2. The holomorphic discrete series; Fock spaces

For $\lambda \in \mathbb{R}$ consider the Fock space $\mathcal{F}_\lambda$ of holomorphic functions on B satisfying

$$\|f\|_\lambda^2 := \int_B |f(z)|^2 \,(1-\|z\|^2)^\lambda \, d\mu(z) < \infty. \tag{4.4}$$

This space is non-trivial for $\lambda > \rho$ ($\rho = n$), since $\mathcal{F}_\lambda$ contains the function which is identically 1 in this case. One has

$$\|1\|_\lambda^2 = \frac{\pi^n}{2(\lambda-1)\cdots(\lambda-n)}. \tag{4.5}$$

Moreover, $\mathcal{F}_\lambda$ is a closed subspace of $L^2(B, d\mu_\lambda)$, hence a Hilbert space, where $d\mu_\lambda(z) = (1-\|z\|^2)^\lambda \, d\mu(z)$. It also has a reproducing kernel, namely

$$E_\lambda(z,w) = \frac{2(\lambda-1)\cdots(\lambda-n)}{\pi^n}\,[1-(w,z)]^{-\lambda}. \tag{4.6}$$

It is also a unitary module for the action of the universal covering group $\widetilde{G}$ of G; for integer λ ($\lambda > \rho$) it is even a G-module: a holomorphic discrete series representation of scalar type. The group G acts by

$$\pi_\lambda(g)f(z) = f\Big(\frac{d\cdot z + c}{\langle b,z\rangle + a}\Big)\,(\langle b,z\rangle + a)^{-\lambda}, \tag{4.7}$$

$g^{-1} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$. Let us denote by $\overline{\mathcal{F}}_\lambda$ the space of complex conjugates of elements in $\mathcal{F}_\lambda$. It consists of anti-holomorphic functions and gives rise to an obvious unitary action $\overline{\pi}_\lambda$ of $\widetilde{G}$ as well. So

$$\overline{\pi}_\lambda(g)f(z) = f\Big(\frac{d\cdot z + c}{\langle b,z\rangle + a}\Big)\,\overline{(\langle b,z\rangle + a)}^{-\lambda} \tag{4.8}$$

if $g^{-1} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$, $f \in \overline{\mathcal{F}}_\lambda$, $\lambda \in \mathbb{Z}$. For $\lambda \in \mathbb{N}$ ($\lambda > \rho$) we get part of the anti-holomorphic discrete series.

4.3. Tensor products

Consider the Hilbert space tensor product $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda$, with $\lambda > \rho$. The group $\widetilde{G}$ acts diagonally. It turns out that we actually have a G-action, which for integer λ is given by

$$g_0 \cdot (f(z) \otimes \overline{g}(w)) = f(g_0^{-1}\cdot z) \otimes \overline{g}(g_0^{-1}\cdot w)\,(a + \langle b,z\rangle)^{-\lambda}\,\overline{(a + \langle b,w\rangle)}^{-\lambda} \tag{4.9}$$

if $g_0^{-1} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$.

Let A_λ denote the bilinear map, defined on tensors in $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda$ by

$$f(z) \otimes \overline{g}(w) \to f(z)\overline{g}(z)\,(1-\|z\|^2)^\lambda. \tag{4.10}$$

Then, restricting A_λ to polynomial functions, A_λ is densely defined with image in $\mathcal{H}_0$. Furthermore, according to J. Repka ([14], Proposition 4.1):

- A_λ has trivial kernel and dense image,
- A_λ intertwines the G-actions on $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda$ and π_0,
- A_λ is closed.

Let $A_\lambda = |A_\lambda A_\lambda^*|^{1/2}\, U_\lambda$ be the polar decomposition of A_λ. Then U_λ is a unitary equivalence between the G-spaces $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda$ and $\mathcal{H}_0 \simeq L^2(G/K)$.

Actually A_λ can be extended to a bounded operator from $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda$ into $\mathcal{H}_0$ with $\|A_\lambda\|^2 \le 1/c_\lambda$, where $c_\lambda = \|1\|_\lambda^2$. Indeed, functions $F(z,w)$, holomorphic in z, anti-holomorphic in w, such that

$$\int_B \int_B |F(z,w)|^2 \, d\mu_\lambda(z)\, d\mu_\lambda(w) < \infty,$$

are general elements in the Hilbert space $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda$. Clearly for such functions F,

$$F(z,t) = \int_B E_\lambda(w,t)\, F(z,w)\, d\mu_\lambda(w), \tag{4.11}$$

therefore,

$$A_\lambda F(z) = \int_B E_\lambda(w,z)\, F(z,w)\, d\mu_\lambda(w) \cdot (1 - \|z\|^2)^\lambda .$$

Hence

$$|A_\lambda F(z)|^2 \le \int_B |E_\lambda(w,z)|^2\, d\mu_\lambda(w) \cdot \int_B |F(z,w)|^2\, d\mu_\lambda(w)\, (1 - \|z\|^2)^{2\lambda} .$$

So

$$\|A_\lambda F\|^2 = \int_B |A_\lambda F(z)|^2\, d\mu(z) \le \frac{1}{c_\lambda} \|F\|_2^2 .$$

4.4. The adjoint of A_λ. The Berezin kernel

Let $F(z,w)$ be holomorphic in z, anti-holomorphic in w, and belonging to $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda$, or $L^2(B \times B, d\mu_\lambda \otimes d\mu_\lambda)$. Let h belong to $L^2(B, d\mu)$. We shall determine an explicit expression for $A_\lambda^* h$. It is clear that $A_\lambda^* h$ is in $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda$, so $A_\lambda^* h(z,w)$ is holomorphic in z and anti-holomorphic in w. One has

$$\begin{aligned}(A_\lambda^* h, F) &= (h,\, A_\lambda F) \\ &= \int_B \int_B h(z)\, E_\lambda(z,w)\, \overline{F}(z,w)\, (1 - \|z\|^2)^\lambda\, d\mu_\lambda(w) d\mu(z).\end{aligned}$$

So $A_\lambda^* h$ is the projection of the function

$$(z,w) \to h(z)\, E_\lambda(z,w)$$

onto $\mathcal{F}_\lambda \widehat{\otimes}_2 \overline{\mathcal{F}}_\lambda$. The above function is in $L^2(B \times B, d\mu_\lambda \otimes d\mu_\lambda)$. The orthogonal projection, call it E, is given by

$$EF(z,w) = \int_B \int_B E_\lambda(w',w)\, E_\lambda(z,z')\, F(z',w')\, d\mu_\lambda(z') d\mu_\lambda(w'). \tag{4.12}$$

Hence

$$A_\lambda^* h(z,w) = \int_B E_\lambda(z,z')\, E_\lambda(z',w)\, h(z')\, d\mu_\lambda(z').$$

Define for $\lambda > \rho$ and $f, g \in \mathcal{H}_0 = L^2(B, d\mu)$:

$$(f,g)_\lambda = (A_\lambda\, A_\lambda^* f, g). \tag{4.13}$$

This form is clearly (strictly) positive-definite. More explicitly:

$$A_\lambda\, A_\lambda^* f(z) =$$

$$\int_B E_\lambda(z,z')\, E_\lambda(z',z)\, f(z')\, d\mu_\lambda(z')\, (1 - \|z\|^2)^\lambda .$$

So $A_\lambda\, A_\lambda^*$ is a kernel operator with kernel

$$B_\lambda(z,z') = c_\lambda^{-2} \left\{ \frac{(1 - \|z\|^2)\,(1 - \|z'\|^2)}{[1 - (z,z')]\,[1 - (z',z)]} \right\}^\lambda . \tag{4.14}$$

This is again the Berezin kernel (up to a factor); it is G-invariant, positive-definite, and defines a bounded Hermitian form on $L^2(G/K)$ for $\lambda > \rho$. Notice that the Berezin kernel is given by

$$\frac{E_\lambda(z,z')\, E_\lambda(z',z)}{E_\lambda(z,z)\, E_\lambda(z',z')}. \tag{4.15}$$

5. MAXIMAL DEGENERATE REPRESENTATIONS OF SL$(n+1,\mathbb{C})$

5.1. Definition of the representations

Let $G = \mathrm{SU}(1,n)$ and $G_c = \mathrm{SL}(n+1,\mathbb{C})$, a complexification. Denote by K_c the subgroup

$$K_c = \mathrm{S}(\mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C}))$$

of G_c and set $U = \mathrm{SU}(n+1)$, $K = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))$. Let $P^{\pm}$ be the two (standard) maximal parabolic subgroups of G_c consisting of upper and lower block matrices respectively:

$$P^+ : \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad P^- : \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \tag{5.1}$$

with $a \in \mathbb{C}^*$, $\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \in K_c$, b a row (column) vector in $\mathbb{C}^n$. For $\mu \in \mathbb{C}$, define the character ω_μ of $P^{\pm}$ by the formula:

$$\omega_\mu(p) = |a|^\mu,$$

where $p \in P^{\pm}$ has one of the forms (5.1). Consider the representations $\pi_\mu^{\pm}$ of G_c induced from $P^{\pm}$:

$$\pi_\mu^{\pm} = \mathrm{Ind}\,\omega_{\mp\mu}. \tag{5.2}$$

Let us describe these representations in the "compact picture". One has the following decompositions:

$$G = UP^+ = UP^-, \tag{5.3}$$

which we call Iwasawa type decompositions. For the corresponding decompositions $g = up$ of an element $g \in G_c$, the factors p and u are defined up to an element of the subgroup $U \cap P^+ = U \cap P^- = U \cap K_c = K$. The coset spaces $G_c/P^{\pm}$ can be identified with the coset space U/K. Set

$$S = \{z \in \mathbb{C}^{n+1} : \|z\|^2 = 1\},$$

which clearly can be identified with $\mathrm{SU}(n+1)/\mathrm{SU}(n)$ via $u \to ue_0$ ($u \in \mathrm{SU}(n+1)$).

Let us denote by $\mathcal{V}$ the vector space of C^∞-functions φ on S satisfying

$$\varphi(\lambda s) = \varphi(s) \tag{5.4}$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

$\mathcal{V}$ can be seen as the representation space of both π_μ^+ and π_μ^-. In fact $\pi_\mu^+ = \pi_\mu^- \circ \tau$ where τ is the Cartan involution of G_c: $\tau(g) = (g^*)^{-1}$.

The group G_c acts on S; denote by $g \cdot s$ ($g \in G_c$, $s \in S$) the action of g on s:

$$g \cdot s = \frac{g(s)}{\|g(s)\|}. \tag{5.5}$$

We have for $\varphi \in \mathcal{V}$:

$$\pi_\mu^-(g)\varphi(s) = \varphi(g^{-1} \cdot s)\,\|g^{-1}(s)\|^\mu. \tag{5.6}$$

In a similar way we have:

$$\pi_\mu^+(g)\varphi(s) = \varphi(\tau(g^{-1}) \cdot s)\,\|\tau(g^{-1})s\|^\mu. \tag{5.7}$$

Let $(\,|\,)$ denote the standard inner product on $L^2(S)$,

$$(\varphi|\psi) = \int_S \varphi(s)\,\overline{\psi(s)}\,ds. \tag{5.8}$$

Here ds is the normalized U-invariant measure on S. This measure ds is transformed by the action of $g \in G_c$ as follows:

$$d\widetilde{s} = \|g(s)\|^{-2(n+1)} ds, \quad \widetilde{s} = g \cdot s. \tag{5.9}$$

It implies that the Hermitian form (5.8) is invariant with respect to the pairs

$$(\pi_\mu^-, \pi^-_{-\bar\mu-2(n+1)}) \quad \text{and} \quad (\pi_\mu^+, \pi^+_{-\bar\mu-2(n+1)}).$$

Therefore, if $\operatorname{Re}\mu = -(n+1)$, then the representations $\pi_\mu^\pm$ are unitarizable, the inner product being (5.8).

5.2. Intertwining operators and irreducibility

It is an interesting problem to determine the complex numbers μ such that $\pi_\mu^\pm$ is irreducible, and in case it is reducible, to obtain the composition series. We will not persue this problem here. We refer to [6] where a similar method is used. It turns out that $\pi_\mu^\pm$ is at least irreducible if $\mu \notin \mathbb{Z}$. Let us turn to intertwining operators.

Define the operator A_μ on $\mathcal{V}$ by the formula

$$A_\mu\varphi(s) = \int_S |(s,t)|^{-\mu-2(n+1)}\varphi(t)\,dt. \tag{5.10}$$

This integral converges absolutely for $\operatorname{Re}\mu < -2n-1$ and can be analytically extended to the whole μ-plane as a mereomorphic function. It is easily checked that A_μ is an intertwining operator

$$A_\mu\,\pi_\mu^\pm(g) = \pi_{\mu'}^\mp(g)\,A_\mu,\ g \in G_c, \tag{5.11}$$

with $\mu' = -\mu - 2(n+1)$.

The operator $A_{-\mu-2(n+1)} \circ A_\mu$ intertwines $\pi_\mu^\pm$ with itself and is therefore a scalar $c(\mu)$, independent of the $\pm$-sign. In general $c(\mu)$ will be a meromorphic function of μ. It can be computed using K-types, see e.g. [6], §1. It turns out that

$$c(\mu) = c(-\mu - 2(n+1)).$$

It turns out, in addition, that only the $\pi^\pm$ with $\operatorname{Re}\mu = -(n+1)$ are unitarizable (see again [6], §1).

5.3. Restriction to G

Consider the diagonal matrix $J = \operatorname{diag}\{1,-1,\ldots,-1\}$. Then

$$G = \{g \in G_c \,:\, g^* = Jg^{-1}J\}. \tag{5.12}$$

So the Cartan involution τ of G_c restricted to G is given by $\tau(g) = JgJ$ $(g \in G)$. Consequently, π_μ^+ is equivalent to π_μ^- on G: the equivalence is given by $\varphi \to E\varphi$ with

$$E\varphi(s) = \varphi(Js) \tag{5.13}$$

for $\varphi \in \mathcal{V}$.

Now consider the action of G on S given by (5.5). There are 3 orbits, given by

$$[s,s] > 0, \qquad [s,s] = 0 \qquad \text{and} \qquad [s,s] < 0. \tag{5.14}$$

All three orbits are invariant under $s \to \lambda s$ with $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Call O_1, O_2, O_3 the corresponding G-orbits on $S/\sim$ where $s \sim s'$ if and only if $s = \lambda s'$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Then we have:

$$O_1 \simeq G/K \qquad \text{via} \qquad g \to g\cdot e_0, \tag{5.15}$$

$$O_2 \simeq G/MAN \qquad \text{via} \qquad g \to g\cdot(e_0 + e_n), \tag{5.16}$$

$$O_3 \simeq G/\mathrm{S}(\mathrm{U}(1,n-1)\times\mathrm{U}(1)) \qquad \text{via} \qquad g \to g\cdot e_n. \tag{5.17}$$

For the subgroup MAN (a minimal parabolic subgroup of G) see section 1.2.

Let φ be a C^∞-function with compact support on $[s,s] \neq 0$, satisfying (5.4). Set

$$\psi(s) = \varphi(s)\,|[s,s]|^{-1/2\mu}. \tag{5.18}$$

Then ψ satifies the same condition (5.4). Moreover,

$$\begin{aligned}\psi(g^{-1}\cdot s) &= \varphi(g^{-1}\cdot s)\,||g^{-1}(s)||^{\mu}\,|[s,s]|^{-1/2\mu}\\ &= \pi_\mu^-(g)\varphi(s)\,|[s,s]|^{-1/2\mu}.\end{aligned}$$

So the linear map $\varphi \rightarrow \psi$ intertwines the restriction of π_μ^- to G with the left regular representation of G on $\mathcal{D}(G/K)$ and $\mathcal{D}(G/H)$ respectively where $H = \mathrm{S}(\mathrm{U}(1, n-1) \times \mathrm{U}(1))$.

A G-invariant measure on $O_1 \cup O_3$ is given by

$$d\nu(s) = \frac{ds}{|[s,s]|^{n+1}}. \tag{5.19}$$

So, if we provide $\mathcal{D}(S)$ with the inner product on $[s,s] \neq 0$ given by

$$\langle \varphi_1, \varphi_2 \rangle = \int_S \varphi_1(s)\, \overline{\varphi_2(s)}\, |[s,s]|^{-\mathrm{Re}\,\mu-(n+1)} ds, \tag{5.20}$$

then π_μ^- becomes unitary, if we restrict it to G. From now on we shall only consider the restriction of π_μ^- to G and call it R_μ. Clearly R_μ is equivalent to $R_{\mu'}$. The intertwinig operator becomes

$$A_\mu \varphi(s) = \int_S |[s,t]|^{-\mu-2(n+1)} \varphi(t) dt. \tag{5.21}$$

Observe that A_μ is defined on $[s,s] > 0$ for all μ, provided φ has compact support in this open set. Then $A_\mu\varphi$ is a C^∞-function on this set (non-necessarily with compact support). On $[s,s] < 0$ one still has to deal with analytic continuation in μ, since convergence of the integral is not garantueed for all μ.

For $\varphi_1, \varphi_2 \in \mathcal{V}$ and $\mu \in \mathbb{R}$, consider the Hermitian form

$$\langle \varphi_1, A_\mu \varphi_2 \rangle = \int_S \int_S |[s,t]|^{-\mu-2(n+1)} \varphi_1(s)\, \overline{\varphi_2(t)} ds dt. \tag{5.22}$$

This form is clearly invariant with respect to R_μ. Applying the linear transformation (5.18) on the open set $[s,s] \neq 0$, we get the following:

$$\langle \psi_1, \psi_2 \rangle = \int_S \int_S \psi_1(s)\, \overline{\psi_2(t)} \left| \frac{[s,s]\,[t,t]}{[s,t]\,[t,s]} \right|^{\frac{\mu}{2}+(n+1)} d\nu(s) d\nu(t). \tag{5.23}$$

Now restrict to $[s,s] > 0$. Then

$$B_\mu(s,t) = \left\{ \frac{[s,s]\,[t,t]}{[s,t]\,[t,s]} \right\}^\mu$$

is the Berezin kernel on $\mathcal{D}(G/K)$, see (3.7).

NOTES

• Canonical representations have been introduced for classical Hermitian symmetric spaces by Berezin and later, in a different context, by Vershik, Gel'fand and Graev for $\mathrm{SL}(2,\mathbb{R})$ (see [1], [16]).

• A more conceptual treatment for Hermitian symmetric spaces in the context of Jordan algebras has recently been given by Upmeier and Unterberger [15].

• An extension to hyperbolic spaces, also for small values of the parameters, and for line bundles over these spaces, is due to Hille and van Dijk [3], [4], [12].

• Canonical representations for para-Hermitian spaces were proposed and introduced by Molchanov [13].

• A thorough treatment of the rank one para-Hermitian space $\mathrm{SL}(n,\mathbb{R})/\mathrm{GL}(n-1,\mathbb{R})$ is due to van Dijk and Molchanov [5], [6].

The generalization to para-Hermitian spaces follows the scheme of section 5., which was proposed by Molchanov.

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